

Generalized Geometric Difference Sequence Spaces and its duals

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ABSTRACT. After introducing the geometric difference sequence spaces in the paper [4], objective of this paper is to introduce the generalized geometric difference sequence spaces $l_{\infty}^G(\Delta_G^m)$, $c^G(\Delta_G^m)$, $c_0^G(\Delta_G^m)$ and to prove that these are Banach spaces. Then we prove some inclusion properties. Also we compute their dual spaces.

Keywords and phrases: Geometric difference; dual space; geometric integers; geometric complex numbers.

AMS subject classification (2000): 26A06, 11U10, 08A05, 46A45.

1. INTRODUCTION

After the introduction to the "Non-Newtonian Calculus" by Grossman and Katz [8] which is also called as multiplicative calculus, various researchers have been developing its dimensions. The operations of multiplicative calculus is called as multiplicative derivative and multiplicative integral. We refer to Grossman and Katz [8], Stanley [18], Bashirov et al. [2, 3], Grossman [7] for elements of multiplicative calculus and its applications. An extension of multiplicative calculus to functions of complex variables is handled in Bashirov and Rıza [1], Uzer [21], Bashirov et al. [3], Çakmak and Başar [5], Tekin and Başar[19], Türkmen and Başar [20]. In [9, 10] Kadak et al studied the new types of sequences spaces over non-Newtonian Calculus and proved some interesting results. Kadak [11] determined the Köthe-Toeplitz duals over non-Newtonian Complex Field.

Geometric calculus is an alternative to the usual calculus of Newton and Leibniz. It provides differentiation and integration tools based on multiplication instead of addition. Every property in Newtonian calculus has an analog in multiplicative calculus. Generally speaking multiplicative calculus is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example for growth related problems, the use of multiplicative calculus is advocated instead of a traditional Newtonian one.

For readers' convenience, it is to be remind that all concepts in classical arithmetic have natural counterparts in α – *arithmetic*. Consider any generator α with range $A \subseteq \mathbb{C}$. By α – *arithmetic*, we mean the arithmetic whose domain is A and operations are defined

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as follows, for $x, y \in A$ and any generator α ,

$$\begin{array}{ll}
 \alpha - \text{addition} & x \dot{+} y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] \\
 \alpha - \text{subtraction} & x \dot{-} y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] \\
 \alpha - \text{multiplication} & x \dot{\times} y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] \\
 \alpha - \text{division} & x \dot{/} y = \alpha[\alpha^{-1}(x) / \alpha^{-1}(y)] \\
 \alpha - \text{order} & x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y).
 \end{array}$$

If we choose “*exp*” as an α -generator defined by $\alpha(z) = e^z$ for $z \in \mathbb{C}$ then $\alpha^{-1}(z) = \ln z$ and α -arithmetic turns out to geometric arithmetic.

$$\begin{array}{llll}
 \alpha - \text{addition} & x \oplus y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] & = e^{(\ln x + \ln y)} = x.y & \text{geometric addition} \\
 \alpha - \text{subtraction} & x \ominus y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] & = e^{(\ln x - \ln y)} = x \div y, y \neq 0 & \text{geometric subtraction} \\
 \alpha - \text{multiplication} & x \odot y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] & = e^{(\ln x \times \ln y)} = x^{\ln y} & \text{geometric multiplication} \\
 \alpha - \text{division} & x \oslash y = \alpha[\alpha^{-1}(x) / \alpha^{-1}(y)] & = e^{(\ln x \div \ln y)} = x^{\frac{1}{\ln y}}, y \neq 1 & \text{geometric division.}
 \end{array}$$

In [20] Türkmen and F. Başar defined the geometric complex numbers $\mathbb{C}(G)$ as follows:

$$\mathbb{C}(G) := \{e^z : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}.$$

Then $(\mathbb{C}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e .

Then for all $x, y \in \mathbb{C}(G)$

- $x \oplus y = xy$
- $x \ominus y = x/y$
- $x \odot y = x^{\ln y} = y^{\ln x}$
- $x \oslash y$ or $\frac{x}{y}G = x^{\frac{1}{\ln y}}, y \neq 1$
- $x_1 \oplus x_2 \oplus \dots \oplus x_n =_G \sum_{i=1}^n x_i = x_1.x_2\dots x_n$
- $x^{2G} = x \odot x = x^{\ln x}$
- $x^{pG} = x^{\ln^{p-1} x}$
- $\sqrt{x}^G = e^{(\ln x)^{\frac{1}{2}}}$
- $x^{-1G} = e^{\frac{1}{\log x}}$
- $x \odot e = x$ and $x \oplus 1 = x$
- $e^n \odot x = x^n = x \oplus x \oplus \dots$ (upto n number of x)
-

$$|x|^G = \begin{cases} x, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \\ \frac{1}{x}, & \text{if } x < 1 \end{cases}$$

Thus $|x|^G \geq 1$.

- $\sqrt{x^{2G}}^G = |x|^G$
- $|e^y|^G = e^{|y|}$
- $|x \odot y|^G = |x|^G \odot |y|^G$
- $|x \oplus y|^G \leq |x|^G \oplus |y|^G$
- $|x \oslash y|^G = |x|^G \oslash |y|^G$
- $|x \ominus y|^G \geq |x|^G \ominus |y|^G$
- $0_G \ominus 1_G \odot (x \ominus y) = y \ominus x$, i.e. in short $\ominus(x \ominus y) = y \ominus x$.

Let l_∞, c and c_0 be the linear spaces of complex bounded, convergent and null sequences, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|.$$

Türkmen and Başar [20] have proved that

$$\omega(G) = \{(x_k) : x_k \in \mathbb{C}(G) \text{ for all } k \in \mathbb{N}\}$$

is a vector space over $\mathbb{C}(G)$ with respect to the algebraic operations \oplus addition and \odot multiplication

$$\begin{aligned} \oplus : \omega(G) \times \omega(G) &\rightarrow \omega(G) \\ (x, y) &\rightarrow x \oplus y = (x_k) \oplus (y_k) = (x_k y_k) \\ \odot : \mathbb{C}(G) \times \omega(G) &\rightarrow \omega(G) \\ (\alpha, y) &\rightarrow \alpha \odot y = \alpha \odot (y_k) = (\alpha^{\ln y_k}), \end{aligned}$$

where $x = (x_k), y = (y_k) \in \omega(G)$ and $\alpha \in \mathbb{C}(G)$. Then

$$l_\infty(G) = \{x = (x_k) \in \omega(G) : \sup_{k \in \mathbb{N}} |x_k|^G < \infty\}$$

$$c(G) = \{x = (x_k) \in \omega(G) : {}_G \lim_{k \rightarrow \infty} |x_k| = 1\}$$

$$c_0(G) = \{x = (x_k) \in \omega(G) : {}_G \lim_{k \rightarrow \infty} x_k = 1\}$$

$$l_p(G) = \{x = (x_k) \in \omega(G) : \sum_{k=0}^{\infty} (|x_k|^G)^{pG} < \infty\}, \text{ where } {}_G \sum \text{ is the geometric sum,}$$

are classical sequence spaces over the field $\mathbb{C}(G)$. Also they have shown that $l_\infty(G)$, $c(G)$ and $c_0(G)$ are Banach spaces with the norm

$$\|x\|^G = \sup_k |x_k|^G, x = (x_1, x_2, x_3, \dots) \in \lambda(G), \lambda \in \{l_\infty, c, c_0\}.$$

Here, ${}_G \lim$ is the geometric limit defined in [20](page no. 27). For the convenience, we denoted $l_\infty(G), c(G), c_0(G)$, respectively as l_∞^G, c^G, c_0^G .

In 1981, Kizmaz [12] introduced the notion of difference sequence spaces using forward difference operator Δ and studied the classical difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$. Following C. Türkmen and F. Başar [20], Kizmaz [12], we defined geometric sequence space in [4] as follows:

$$l_\infty^G(\Delta_G) = \{x = (x_k) \in \omega(G) : \Delta_G x \in l_\infty^G\}, \text{ where } \Delta_G x = x_k \ominus x_{k+1}.$$

where $\Delta_G x = (\Delta_G x_k) = (x_k \ominus x_{k+1})$. Then we introduced some theorems, definitions and basic results as follows:

Theorem 1.1. *The space $l_\infty^G(\Delta_G)$ is a normed linear space w.r.t. the norm*

$$(1.1) \quad \|x\|_{\Delta_G}^G = |x_1|^G \oplus \|\Delta_G x\|_\infty^G.$$

Theorem 1.2. *The space $l_\infty^G(\Delta_G)$ is a Banach space w.r.t. the norm $\|\cdot\|_{\Delta_G}^G$.*

Remark 1.1. The spaces

- (a) $c^G(\Delta_G) = \{(x_k) \in w(G) : \Delta_G x_k \in c^G\}$
- (b) $c_0^G(\Delta_G) = \{(x_k) \in w(G) : \Delta_G x_k \in c_0^G\}$

are Banach spaces with respect to the norm $\|\cdot\|_{\Delta_G}^G$. Also these spaces are BK-spaces.

Lemma 1.3. *The following conditions (a) and (b) are equivalent:*

- (a) $\sup_k |x_k \ominus x_{k+1}|^G < \infty$ i.e. $\sup_k |\Delta_G x_k|^G < \infty$;
 (b)(i) $\sup_k e^{k-1} \odot |x_k|^G < \infty$ and
 (ii) $\sup_k |x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1}|^G < \infty$.

Lemma 1.4.

$$\text{If } \sup_n \left| {}_G \sum_{v=1}^n c_v \right|^G \leq \infty \text{ then } \sup_n \left(p_n \odot \left| {}_G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} G \right|^G \right) < \infty.$$

Corollary 1.5. *Let (p_n) be monotonically increasing. If*

$$\sup_n \left| {}_G \sum_{v=1}^n p_v \odot a_v \right|^G < \infty \text{ then } \sup_n \left| p_n \odot {}_G \sum_{k=n+1}^{\infty} a_k \right|^G < \infty.$$

Corollary 1.6.

$$\text{If } {}_G \sum_{k=1}^{\infty} p_k \odot a_k \text{ is convergent then } \lim_n p_n \odot {}_G \sum_{k=n+1}^{\infty} a_k = 1.$$

Corollary 1.7.

$${}_G \sum_{k=1}^{\infty} e^k \odot a_k \text{ is convergent iff } {}_G \sum_{k=1}^{\infty} R_k \text{ is convergent with } e^n \odot R_n = O(e), \text{ where}$$

$$R_n = {}_G \sum_{k=n+1}^{\infty} a_k.$$

Definition 1.1. [6, 13, 14, 15] If X is a sequence space, we define

- (i) $X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\}$;
 (ii) $X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X\}$;
 (iii) $X^\gamma = \{a = (a_k) : \sup_n |\sum_{k=1}^n a_k x_k| < \infty, \text{ for each } x \in X\}$.

Theorem 1.8.

- (i) If $D_1 = \left\{ a = (a_k) : {}_G \sum_{k=1}^{\infty} e^k \odot |a_k|^G < \infty \right\}$ then $(sl_\infty^G(\Delta_G))^\alpha = D_1$.
 (ii) If $D_2 = \left\{ a = (a_k) : {}_G \sum_{k=1}^{\infty} e^k \odot a_k \text{ is convergent with } {}_G \sum_{k=1}^{\infty} |R_k|^G < \infty \right\}$.

Then $(sl_\infty^G(\Delta_G))^\beta = D_2$.

- (iii) If $D_3 = \left\{ a = (a_k) : \sup_n \left| {}_G \sum_{k=1}^n e^k \odot a_k \right|^G < \infty, {}_G \sum_{k=1}^{\infty} |R_k|^G < \infty \right\}$.

Then $(sl_\infty^G(\Delta_G))^\gamma = D_3$.

Following Kizmaz, generalized sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ were introduced by Mikail Et and Rifat Çolak [16].

2. MAIN RESULTS

Following Mikail Et and Rifat Çolak [16, 17] and our own paper[4], now we define the following new sequence spaces

$$\begin{aligned} l_\infty^G(\Delta_G^m) &= \{x = (x_k) : \Delta_G^m x \in lG_\infty\}, \\ c^G(\Delta_G^m) &= \{x = (x_k) : \Delta_G^m x \in c^G\}, \\ l_0^G(\Delta_G^m) &= \{x = (x_k) : \Delta_G^m x \in c_0^G\}. \end{aligned}$$

where $m \in \mathbb{N}$ and

$$\begin{aligned} \Delta_G^0 x &= (x_k) \\ \Delta_G x &= (\Delta_G x_k) = (x_k \ominus x_{k+1}) \\ \Delta_G^2 x &= (\Delta_G^2 x_k) = (\Delta_G x_k \ominus \Delta_G x_{k+1}) \\ &= (x_k \ominus x_{k+1} \ominus x_{k+1} \oplus x_{k+2}) \\ &= (x_k \ominus e^2 \odot x_{k+1} \oplus x_{k+1}) \\ \Delta_G^3 x &= (\Delta_G^3 x_k) = (\Delta_G^2 x_k \ominus \Delta_G^2 x_{k+1}) \\ &= (x_k \ominus e^3 \odot x_{k+1} \oplus e^3 \odot x_{k+1} \ominus x_{k+3}) \\ &\dots\dots\dots \\ \Delta_G^m x &= (\Delta_G^m x_k) = (\Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1}) \\ &= \left({}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{(m)} \odot x_{k+v} \right), \text{ with } (\ominus e)^{0G} = e. \end{aligned}$$

Then it can be easily proved that $l_\infty^G(\Delta_G^m)$, $c_\infty^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are normed linear spaces with norm

$$\|x\|_{\Delta_G}^G = {}_G \sum_{i=1}^m |x_i|^G \oplus \|\Delta_G^m x\|_\infty^G.$$

Note: Throughout this paper often we write ${}_G \sum_k$ instead of ${}_G \sum_{k=1}^\infty$ and \lim_k instead of $\lim_{n \rightarrow \infty}$.

Definition 2.1 (Geometric Associative Algebra). An associative algebra is a vector space $A \subset \mathbb{R}(G)$, equipped with a bilinear map (called multiplication)

$$\begin{aligned} \odot : A \times A &\rightarrow A \\ (a, b) &\rightarrow a \odot b \end{aligned}$$

which is associative, i.e.

$$(a \odot b) \odot c = a \odot (b \odot c) \forall a, b, c \in A.$$

An algebra is commutative if $a \odot b = b \odot a$ for all $a, b \in A$. An algebra is unital if there exists a unique $e \in A$ such that $e \odot a = a \odot e = a$ for all $a \in A$. A subalgebra of the algebra A is a subspace B that is closed under multiplication, i.e. $a \odot b \in A$ for all $a, b \in B$.

Definition 2.2 (Geometric Normed Algebra). A normed algebra is a normed space $A \subset \omega(G)$ that is also an associative algebra, such that the norm is submultiplicative: $\|a \odot b\|^G \leq \|a\|^G \odot \|b\|^G$ for all $a, b \in A$. A geometric algebra is a complete normed algebra, i.e., a normed algebra which is also a Banach space with respect to its norm.

It is to be noted that the submultiplicativity of the norm means that multiplication in normed algebras is jointly continuous, i.e. if $a_n \xrightarrow{G} a$ and $b_n \xrightarrow{G} b$ then (a_n) is bounded and

$$\begin{aligned} \|a_n \odot b_n \ominus a \odot b\|^G &= \|a_n \odot (b_n \ominus b) \oplus (a_n \ominus a) \odot b\|^G \\ &\leq \|a_n\|^G \odot \|b_n \ominus b\|^G \oplus \|a_n \ominus a\|^G \odot \|b\|^G \\ &\leq \sup \left\{ \|a_n\|^G \right\} \odot \|b_n \ominus b\|^G \oplus \|b\|^G \odot \|a_n \ominus a\|^G \xrightarrow{G} 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

Definition 2.3 (Geometric Sequence Algebra). A geometric sequence space $E(G)$ is said to be sequence algebra if $x \odot y \in E(G)$ for $x = (x_k), y = (y_k) \in E(G)$. i.e. $E(G)$ is closed under the geometric multiplication \odot defined by

$$\begin{aligned} \odot : E(G) \times E(G) &\rightarrow E(G) \\ (x, y) &\rightarrow x \odot y = (x_k) \odot (y_k) = (x_k^{\ln y_k}) \end{aligned}$$

for any two sequences $x = (x_k), y = (y_k) \in E$.

Since $\omega(G)$ is closed under geometric multiplication \odot , hence, $\omega(G)$ is a sequence algebra. Also sequence algebra $\omega(G)$ is unital as $\|e_G\|^G = e$, where $e_G = (e, e, e, \dots) \in \omega(G)$.

Definition 2.4 (Continuous Dual Space). If X is a normed space, a linear map $f : X \rightarrow \mathbb{R}(G)$ is called linear functional. f is called continuous linear functional or bounded linear functional if $\|f\|^G < \infty$, where

$$\|f\|^G = \sup \left\{ |f(x)|^G : \|x\|^G \leq e, \text{ for all } x \in X \right\}$$

Let X^* be the collection of all bounded linear functionals on X . If $f, g \in X^*$ and $\alpha \in \mathbb{R}(G)$, we define $(\alpha \odot f \oplus g)(x) = \alpha \odot f(x) \oplus g(x)$; X^* is called the continuous dual space of X .

Theorem 2.1. The sequence spaces $l_\infty^G(\Delta_G^m)$, $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are Banach spaces with the norm

$$\|x\|_{\Delta_G}^G =_G \sum_{i=1}^m |x_i|^G \oplus \|\Delta_G^m x\|_\infty^G.$$

Proof. Let (x_n) be a Cauchy sequence in $l_\infty^G(\Delta_G^m)$, where $x_n = (x_i^{(n)}) = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots)$ for $n \in \mathbb{N}$ and $x_k^{(n)}$ is the k^{th} coordinate of x_n . Then

$$\begin{aligned} \|x_n \ominus x_l\|_{\Delta_G}^G &=_G \sum_{i=1}^m \left| x_i^{(n)} \ominus x_i^{(l)} \right|^G \oplus \|\Delta_G^m(x_n \ominus x_l)\|_\infty^G \\ (2.1) \quad &=_G \sum_{i=1}^m \left| x_i^{(n)} \ominus x_i^{(l)} \right|^G \oplus \sup_k |\Delta_G^m(x_n \ominus x_l)|^G \rightarrow 1 \text{ as } l, n \rightarrow \infty. \end{aligned}$$

Hence we obtain

$$|x_k^{(n)} \ominus x_k^{(l)}|^G \rightarrow 1$$

as $n, l \rightarrow \infty$ and for each $k \in \mathbb{N}$. Therefore $(x_k^{(n)}) = (x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, \dots)$ is a Cauchy sequence in $\mathbb{C}(G)$. Since $\mathbb{C}(G)$ is complete, $(x_k^{(n)})$ is convergent.

Suppose $\lim_n x_k^{(n)} = x_k$, for each $k \in \mathbb{N}$. Since (x_n) is a Cauchy sequence, for each $\epsilon > 1$, there exists $N = N(\epsilon)$ such that $\|x_n \ominus x_l\|_{\Delta_G}^G < \epsilon$ for all $n, l \geq N$. Hence from (2.1)

$$_G \sum_{i=1}^m \left| x_i^{(n)} \ominus x_i^{(l)} \right|^G < \epsilon \text{ and } \left| _G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{(m)} \odot (x_{k+v}^{(n)} \ominus x_{k+v}^{(l)}) \right|^G < \epsilon$$

for all $k \in \mathbb{N}$ and $n, l \geq N$. So we have

$$\lim_l {}_G \sum_{i=1}^m \left| x_i^{(n)} \ominus x_i^{(l)} \right|^G = {}_G \sum_{i=1}^m \left| x_i^{(n)} \ominus x_i \right|^G < \epsilon$$

$$\text{and } \lim_l \left| {}_G \Delta_G^m(x_k^{(n)} \ominus x_k^{(l)}) \right|^G = \left| {}_G \Delta_G^m(x_k^{(n)} \ominus x_k) \right|^G < \epsilon \quad \forall n \geq N.$$

This implies $\|x_n \ominus x\|_{\Delta_G}^G < \epsilon^2 \quad \forall n \geq N$, that is $x_n \xrightarrow{G} x$ as $n \rightarrow \infty$, where $x = (x_k)$. Now we have to show that $x \in l_\infty^G(\Delta_G^m)$. We have

$$\begin{aligned} |\Delta_G^m x_k|^G &= \left| {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{(m)_v} \odot x_{k+v} \right|^G \\ &= \left| {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{(m)_v} \odot (x_{k+v} \ominus x_{k+v}^N \oplus x_{k+v}^N) \right|^G \\ &\leq \left| {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{(m)_v} \odot (x_{k+v}^N \ominus x_{k+v}) \right|^G \oplus \left| {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{(m)_v} \odot x_{k+v}^N \right|^G \\ &\leq \|x^N \ominus x\|_{\Delta_G}^G \oplus |\Delta_G^m x_k^N|^G = O(e). \end{aligned}$$

Therefore we obtain $x \in l_\infty^G(\Delta_G^m)$. Hence $l_\infty^G(\Delta_G^m)$ is a Banach space. \square

It can be shown that $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are closed subspaces of $l_\infty^G(\Delta_G^m)$. Therefore these sequence spaces are Banach spaces with the same norm defined for $l_\infty^G(\Delta_G^m)$, above.

Now we give some inclusion relations between these sequence spaces.

Lemma 2.2.

- (i) $c_0^G(\Delta_G^m) \subsetneq c_0^G(\Delta_G^{m+1})$;
- (ii) $c^G(\Delta_G^m) \subsetneq c^G(\Delta_G^{m+1})$;
- (iii) $l_\infty^G(\Delta_G^m) \subsetneq l_\infty^G(\Delta_G^{m+1})$.

Proof. (i) Let $x \in c_0^G(\Delta_G^m)$. Since

$$\begin{aligned} |\Delta_G^{m+1} x_k|^G &= |\Delta_G^m x_k \ominus \Delta_G^m x_{k+1}|^G \\ &\leq |\Delta_G^m x_k|^G \oplus |\Delta_G^m x_{k+1}|^G \rightarrow 1 \text{ as } k \rightarrow \infty. \end{aligned}$$

\therefore we obtain $x \in c_0^G(\Delta_G^{m+1})$. Thus $c_0^G(\Delta_G^m) \subset c_0^G(\Delta_G^{m+1})$.

This inclusion is strict. For let

$$x = (e^{k^m}) = (e, e^{2^m}, e^{3^m}, e^{4^m}, \dots, e^{k^m}, \dots).$$

Then $x \in c_0^G(\Delta_G^{m+1})$ as $(m+1)^{\text{th}}$ geometric difference of e^{k^m} is 1 (geometric zero). But $x \notin c_0^G(\Delta_G^m)$ as m^{th} geometric difference of e^{k^m} is a constant. Hence the inclusion is strict.

The proofs of (ii) and (iii) are similar to that of (i). \square

Lemma 2.3.

- (i) $c_0^G(\Delta_G^m) \subsetneq c^G(\Delta_G^m)$;
- (ii) $c^G(\Delta_G^m) \subsetneq l_\infty^G(\Delta_G^m)$.

Proofs are similar to that of Lemma (2.2).

Furthermore, since the sequence spaces $l_\infty^G(\Delta_G^m)$, $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are Banach spaces with continuous coordinates, that is, $\|x_n \ominus x\|_{\Delta_G}^G \rightarrow 1$ implies $\left| x_k^{(n)} \ominus x_k \right|^G \rightarrow 1 \forall k \in \mathbb{N}$ as $n \rightarrow \infty$, these are also BK-spaces.

Remark 2.1. It can be easily proved that c_0^G is a sequence algebra. But in general, $l_\infty^G(\Delta_G^m)$, $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are not sequence algebra. For let $x = (e^k)$, $y = (e^{k^{m-1}})$. Clearly $x, y \in c_0^G(\Delta_G^m)$. But

$$x \odot y = \left(e^k \odot e^{k^{m-1}} \right) = (e^{k^m}) \notin c_0^G(\Delta_G^m) \text{ for } m \geq 2,$$

since m^{th} geometric difference of e^{k^m} is constant.

Let us define the operator

$$D : l_\infty^G(\Delta_G^m) \rightarrow l_\infty^G(\Delta_G^m) \text{ as}$$

$Dx = (1, 1, 1, \dots, 1, x_{m+1}, x_{m+2}, \dots)$, where $x = (x_1, x_2, x_3, \dots, x_m, x_{m+1}, \dots) \in l_\infty^G(\Delta_G^m)$. It is trivial that D is a bounded linear operator on $l_\infty^G(\Delta_G^m)$. Furthermore, the set

$$D[l_\infty^G(\Delta_G^m)] = D l_\infty^G(\Delta_G^m) = \{x = (x_k) : x \in l_\infty^G(\Delta_G^m), x_1 = x_2 = \dots = x_m = 1\}$$

is a subspace of $l_\infty^G(\Delta_G^m)$ and

$$\begin{aligned} \|x\|_{\Delta_G}^G &= |x_1|^G \oplus |x_2|^G \oplus \dots \oplus |x_m|^G \oplus \|\Delta_G^m x\|_\infty^G \\ &= 1 \oplus 1 \oplus \dots \oplus 1 \oplus \|\Delta_G^m x\|_\infty^G \\ &= \|\Delta_G^m x\|_\infty^G \\ \therefore \|x\|_{\Delta_G}^G &= \|\Delta_G^m x\|_\infty^G \text{ in } D l_\infty^G(\Delta_G^m). \end{aligned}$$

Now let us define

$$(2.2) \quad \begin{aligned} \Delta^m : D l_\infty^G(\Delta_G^m) &\rightarrow l_\infty^G \\ \Delta_G^m x &= y = (\Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1}). \end{aligned}$$

Δ_G^m is a linear homomorphism: Let $x, y \in D l_\infty^G(\Delta_G^m)$. Then

$$\begin{aligned} \Delta_G^m(x_k \oplus y_k) &= {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot (x_k \oplus y_k) \\ &= {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot x_k \oplus {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot y_k \\ &= \Delta_G^m x_k \oplus \Delta_G^m y_k \\ \therefore \Delta_G^m(x \oplus y) &= \Delta_G^m x \oplus \Delta_G^m y. \text{ For } \alpha \in \mathbb{C}(G) \\ \Delta_G^m(\alpha \odot x) &= (\Delta_G^m \alpha \odot x_k) \\ &= \left({}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot \alpha \odot x_k \right) \\ &= \left(\alpha \odot {}_G \sum_{v=0}^m (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot x_k \right) \\ &= \alpha \odot \Delta_G^m x. \end{aligned}$$

This implies that Δ_G^m is a linear homomorphism. Hence $Dl_\infty^G(\Delta_G^m)$ and l_∞^G are equivalent as topological spaces [15]. Δ_G^m and $(\Delta_G^m)^{-1}$ are norm preserving and

$$\|\Delta_G^m\|_\infty^G = \|(\Delta_G^m)^{-1}\|_\infty^G = e.$$

Let $[l_\infty^G]'$ and $[Dl_\infty^G(\Delta_G^m)]'$ denote the continuous duals of l_∞^G and $Dl_\infty^G(\Delta_G^m)$, respectively. It can be shown that

$$\begin{aligned} s : [Dl_\infty^G(\Delta_G^m)]' &\rightarrow [l_\infty^G]' \\ f_\Delta &\rightarrow f_\Delta \circ (\Delta_G^m)^{-1} = f \end{aligned}$$

is a linear isometry. So $[Dl_\infty^G(\Delta_G^m)]'$ is equivalent to $[l_\infty^G]'$.

In the same way, it can be shown that $Dc^G(\Delta_G^m)$ and $Dc_0^G(\Delta_G^m)$ are equivalent as topological space to c^G and c_0^G , respectively. Also

$$[Dc^G(\Delta_G^m)]' \cong [Dc_0^G(\Delta_G^m)]' \cong l_1^G,$$

where $l_1^G = \{x = (x_k) : \sum_k |x_k|^G < \infty\}$.

3. DUAL SPACES OF $l_\infty^G(\Delta_G^m)$ AND $c^G(\Delta_G^m)$

In this section we construct the α -dual spaces of $l_\infty^G(\Delta_G^m)$ and $c^G(\Delta_G^m)$. Also we show that these spaces are not perfect spaces.

Lemma 3.1. *The following conditions (a) and (b) are equivalent:*

- (a) $\sup_k |x_k \ominus x_{k+1}|^G < \infty$ i.e. $\sup_k |\Delta_G x_k|^G < \infty$;
- (b) (i) $\sup_k e^{k-1} \odot |x_k|^G < \infty$ and
(ii) $\sup_k |x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1}|^G < \infty$.

Proof. Let (a) be true i.e. $\sup_k |x_k \ominus x_{k+1}|^G < \infty$.

$$\begin{aligned} \text{Now } |x_1 \ominus x_{k+1}|^G &= \left| {}_G \sum_{v=1}^k (x_v \ominus x_{v+1}) \right|^G \\ &= \left| {}_G \sum_{v=1}^k \Delta_G x_v \right|^G \\ &\leq {}_G \sum_{v=1}^k |\Delta_G x_v|^G = O(e^k) \end{aligned}$$

and $|x_k|^G = |x_1 \ominus x_1 \oplus x_{k+1} \oplus x_k \ominus x_{k+1}|^G$

$$\leq |x_1|^G \oplus |x_1 \ominus x_{k+1}|^G \oplus |x_k \ominus x_{k+1}|^G = O(e^k).$$

This implies that $\sup_k e^{k^{-1}} \odot |x_k|^G < \infty$. This completes the proof of $b(i)$.
Again

$$\begin{aligned}
\sup_k \left| x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G &= \left| \left\{ e^{(k+1)} \odot e^{(k+1)^{-1}} \right\} \odot x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\
&= \left| \left\{ (e^k \oplus e) \odot e^{(k+1)^{-1}} \right\} \odot x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\
&= \left| \left\{ e^{k(k+1)^{-1}} \odot x_k \oplus e^{(k+1)^{-1}} \odot x_k \right\} \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\
&= \left| \left\{ e^{k(k+1)^{-1}} \odot (x_k \ominus x_{k+1}) \right\} \oplus \left\{ e^{(k+1)^{-1}} \odot x_k \right\} \right|^G \\
&\leq e^{k(k+1)^{-1}} \odot |x_k \ominus x_{k+1}|^G \oplus e^{(k+1)^{-1}} \odot |x_k|^G \\
&= O(e).
\end{aligned}$$

Therefore $\sup_k |x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1}|^G < \infty$. This completes the proof of $b(ii)$.
Conversely let (b) be true. Then

$$\begin{aligned}
\left| x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G &= \left| e^{(k+1)(k+1)^{-1}} \odot x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G \\
&\geq e^{k(k+1)^{-1}} \odot |x_k \ominus x_{k+1}|^G \ominus e^{(k+1)^{-1}} \odot |x_k|^G
\end{aligned}$$

$$\text{i.e. } e^{k(k+1)^{-1}} \odot |x_k \ominus x_{k+1}|^G \leq e^{(k+1)^{-1}} \odot |x_k|^G \oplus \left| x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1} \right|^G.$$

Thus $\sup_k |x_k \ominus x_{k+1}|^G < \infty$ as $b(i)$ and $b(ii)$ hold. □

Corollary 3.2. *The following conditions (a) and (b) are equivalent*

$$\begin{aligned}
(a) \sup_k \left| \Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1} \right|^G &< \infty; \\
(b)(i) \sup_k e^{k^{-1}} \odot \left| \Delta_G^{m-1} x_k \right|^G &< \infty \\
(ii) \sup_k \left| \Delta_G^{m-1} x_k \ominus e^{k(k+1)^{-1}} \odot \Delta_G^{m-1} x_{k+1} \right|^G &< \infty.
\end{aligned}$$

Proof. By putting $\Delta_G^{m-1} x_k$ instead of x_k in Lemma (3.1), results are obvious. □

Lemma 3.3.

$$\sup_k e^{k^{-i}} \odot |\Delta_G x_k|^G < \infty \text{ implies } \sup_k e^{-(i+1)} \odot |x_k|^G < \infty \quad \forall i \in \mathbb{N}.$$

Proof. For $i = 1$ it is obvious from the Lemma (3.1). Let the result be true for $i = n$. i.e. $\sup_k e^{k^{-n}} \odot |\Delta_G x_k|^G < \infty$. Then

$$\begin{aligned}
|x_k \ominus x_{k+1}|^G &= |_G \sum_{v=1}^k \Delta_G x_k|^G \\
&\leq |_G \sum_{v=1}^k |\Delta_G x_k|^G = O\left((e^{k^n})^k\right) = O\left(e^{k^{(n+1)}}\right), \text{ as } \sup_k e^{k^{-n}} \odot |\Delta_G x_k|^G < \infty
\end{aligned}$$

$$\begin{aligned}
\text{and } |x_k|^G &= |x_k \oplus x_1 \ominus x_1 \oplus x_{k+1} \ominus x_{k+1}|^G \\
&\leq |x_1|^G \oplus |x_1 \ominus x_{k+1}|^G \oplus |x_k \ominus x_{k+1}|^G = O\left(e^{k^{(n+1)}}\right).
\end{aligned}$$

From this we obtain, $\sup_k e^{k^{-(n+1)}} \odot |x_k|^G < \infty$. Thus $\sup_k e^{k^{-(i+1)}} \odot |x_k|^G < \infty \quad \forall i \in \mathbb{N}$. □

Lemma 3.4.

$$\begin{aligned} \sup_k e^{k-i} \odot |\Delta_G^{m-1} x_k|^G &< \infty \text{ implies} \\ \sup_k e^{-(i+1)} \odot |\Delta_G^{m-(i+1)} x_k|^G &< \infty \quad \forall i, m \in \mathbb{N} \text{ and } 1 \leq i < m. \end{aligned}$$

Proof. Putting $\Delta_G^{m-i} x_k$ instead of $\Delta_G x_k$ in Lemma (3.3), the result is immediate. \square

Corollary 3.5. $\sup_k e^{k-1} \odot |\Delta_G^{m-1} x_k| < \infty$ implies $\sup_k e^{k-m} \odot |x_k| < \infty$.

Proof. In Lemma (3.4) putting $i = 1$, we get

$$\sup_k e^{k-1} \odot |\Delta_G^{m-1} x_k|^G < \infty \quad \Rightarrow \quad \sup_k e^{k-2} \odot |\Delta_G^{m-2} x_k|^G < \infty$$

Similarly,

$$\sup_k e^{k-2} \odot |\Delta_G^{m-2} x_k|^G < \infty \quad \Rightarrow \quad \sup_k e^{k-3} \odot |\Delta_G^{m-3} x_k|^G < \infty.$$

Continuing the process we get

$$\sup_k e^{k-(m-1)} \odot |\Delta_G^1 x_k|^G < \infty \quad \Rightarrow \quad \sup_k e^{k-m} \odot |\Delta_G^0 x_k|^G < \infty$$

$$\text{Thus } \sup_k e^{k-1} \odot |\Delta_G^{m-1} x_k| < \infty \quad \Rightarrow \quad \sup_k e^{k-m} \odot |x_k| < \infty.$$

\square

Corollary 3.6. If $x \in l_\infty^G(\Delta_G^m)$ then $\sup_k e^{k-m} \odot |x_k|^G < \infty$.

Proof.

$$\begin{aligned} x \in l_\infty^G(\Delta_G^m) &\Rightarrow \Delta_G^m x \in l_\infty^G \\ &\Rightarrow \sup_k |\Delta_G^m x_k|^G < \infty \\ &\Rightarrow \sup_k |\Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1}|^G < \infty \\ &\Rightarrow \sup_k e^{k-1} \odot |\Delta_G^m x_k|^G < \infty \quad \text{by Corollary (3.2)} \\ &\Rightarrow \sup_k e^{k-m} \odot |x_k|^G < \infty \quad \text{by Corollary (3.5)}. \end{aligned}$$

\square

4. α -, β -, γ - DUALS

Definition 4.1. [6, 13, 14, 15] If X is a sequence space, it is defined that

- (i) $X^\alpha = \{a = (a_k) : \sum_{k=1}^\infty |a_k x_k| < \infty, \text{ for each } x \in X\};$
- (ii) $X^\beta = \{a = (a_k) : \sum_{k=1}^\infty a_k x_k \text{ is convergent, for each } x \in X\};$
- (iii) $X^\gamma = \{a = (a_k) : \sup_n |\sum_{k=1}^n a_k x_k| < \infty, \text{ for each } x \in X\}.$

Then X^α , X^β and X^γ are called α -dual (or Köthe-Toeplitz dual), β -dual (or generalized Köthe-Toeplitz dual) and γ -dual spaces of X , respectively. Then $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\dagger \subset X^\dagger$, for $\dagger = \alpha, \beta$ or γ . It is clear that $X \subset (X^\alpha)^\alpha = X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$ then X is called α -space. α -space is also called a Köthe space or a perfect sequence space.

Then we defined and have proved that[4]
 $(sl_\infty^G(\Delta_G))^\alpha = \{a = (a_k) : {}_G \sum_{k=1}^\infty e^k \odot |a_k|^G < \infty\}$

$(sl_\infty^G(\Delta_G))^\beta = \{a = (a_k) : {}_G \sum_{k=1}^\infty e^k \odot a_k \text{ is convergent with } {}_G \sum_{k=1}^\infty |R_k|^G < \infty\}$
 $(sl_\infty^G(\Delta_G))^\gamma = \{a = (a_k) : \sup_n |{}_G \sum_{k=1}^n e^k \odot a_k|^G < \infty, {}_G \sum_{k=1}^\infty |R_k|^G < \infty\}$
 where $R_k = {}_G \sum_{n=k+1}^\infty a_n$ and $s : l_\infty^G(\Delta_G) \rightarrow l_\infty^G(\Delta_G), x \rightarrow sx = y = (1, x_2, x_3, \dots)$.

Lemma 4.1. *Let $U_1 = \{a = (a_k) : {}_G \sum_k e^{k^m} \odot |a_k|^G < \infty\}$. Then $[Dl_\infty^G(\Delta_G^m)]^\alpha = U_1$.*

Proof. Let $a \in U_1$, then using Corollary (3.2) for $x \in Dl_\infty^G(\Delta_G^m)$, we have
 ${}_G \sum_k |a_k \odot x_k|^G = {}_G \sum_k \{e^{k^m} \odot |a_k|^G\} \odot \{e^{k^{-m}} \odot |x_k|^G\} < \infty$ by Corollary (3.5).

This implies that $a \in [Dl_\infty^G(\Delta_G^m)]^\alpha$. Therefore

$$(4.1) \quad U_1 \subseteq [Dl_\infty^G(\Delta_G^m)]^\alpha.$$

Conversely, let $a \in [Dl_\infty^G(\Delta_G^m)]^\alpha$. Then ${}_G \sum_k |a_k \odot x_k|^G < \infty$ (by definition of α -dual) for $x \in Dl_\infty^G(\Delta_G^m)$. So we take

$$(4.2) \quad x_k = \begin{cases} 1, & \text{if } k \leq m \\ e^{k^m}, & \text{if } k > m \end{cases}$$

Then $x = (1, 1, 1, \dots, 1, e^{(m+1)^m}, e^{(m+2)^m}, \dots) \in Dl_\infty^G(\Delta_G^m)$. Therefore

$$\begin{aligned} {}_G \sum_{k=1}^\infty e^{k^m} \odot |a_k|^G &= {}_G \sum_{k=1}^m e^{k^m} \odot |a_k|^G \oplus {}_G \sum_{k=m+1}^\infty e^{k^m} \odot |a_k|^G \\ &= {}_G \sum_{k=1}^m e^{k^m} \odot |a_k|^G \oplus {}_G \sum_{k=1}^\infty |a_k \odot x_k|^G < \infty \end{aligned}$$

since $a_k \odot x_k = 1$ (the geometric zero) for $k = 1, 2, \dots, m$.

Therefore $a \in U_1$. This implies

$$(4.3) \quad [Dl_\infty^G(\Delta_G^m)]^\alpha \subseteq U_1.$$

Then from (4.1) and (4.3), we get

$$[Dl_\infty^G(\Delta_G^m)]^\alpha = U_1.$$

□

Lemma 4.2.

$$[Dl_\infty^G(\Delta_G^m)]^\alpha = [Dc^G(\Delta_G^m)]^\alpha.$$

Proof. Since $Dc^G(\Delta_G^m) \subseteq Dl_\infty^G(\Delta_G^m)$, hence $[Dl_\infty^G(\Delta_G^m)]^\alpha \subseteq [Dc^G(\Delta_G^m)]^\alpha$.

Again let $a \in [Dc^G(\Delta_G^m)]^\alpha$. Then ${}_G \sum_k |a_k \odot x_k|^G < \infty$ for each $x \in Dc^G(\Delta_G^m)$. If we take $x = (x_k)$ which is defined in (4.2), we get

$${}_G \sum_k e^{k^m} \odot |a_k|^G = {}_G \sum_{k=1}^m e^{k^m} \odot |a_k|^G \oplus {}_G \sum_k |a_k \odot x_k|^G < \infty.$$

This implies that $a \in [Dl_\infty^G(\Delta_G^m)]^\alpha$. Thus

$$[Dl_\infty^G(\Delta_G^m)]^\alpha = [Dc^G(\Delta_G^m)]^\alpha.$$

□

Lemma 4.3.

$$(i) \quad [l_\infty^G(\Delta_G^m)]^\alpha = [Dl_\infty^G(\Delta_G^m)]^\alpha.$$

$$(ii) \quad [c^G(\Delta_G^m)]^\alpha = [Dc^G(\Delta_G^m)]^\alpha.$$

Proof. (i) Since $Dl_\infty^G(\Delta_G^m) \subseteq l_\infty^G(\Delta_G^m)$, so $[l_\infty^G(\Delta_G^m)]^\alpha \subseteq [Dl_\infty^G(\Delta_G^m)]^\alpha$.

Let $a \in [Dl_\infty^G(\Delta_G^m)]^\alpha$ and $x \in l_\infty^G(\Delta_G^m)$. From Corollary (3.6), we have

$${}_G \sum_k |a_k \odot x_k|^G = {}_G \sum_k e^{k^m} \odot |a_k|^G \odot (e^{k^{-m}} \odot |x_k|^G) < \infty.$$

Hence $a \in [l_\infty^G(\Delta_G^m)]^\alpha$.

(ii) $Dc^G(\Delta_G^m) \subseteq c^G(\Delta_G^m)$ implies $[c^G(\Delta_G^m)]^\alpha \subseteq [Dc^G(\Delta_G^m)]^\alpha$.

Let $a \in [Dc^G(\Delta_G^m)]^\alpha$ and $x \in c^G(\Delta_G^m)$. From Corollary (3.6), we have

$${}_G \sum_k |a_k \odot x_k|^G = {}_G \sum_k e^{k^m} \odot |a_k|^G \odot (e^{k^{-m}} \odot |x_k|^G) < \infty$$

for $x \in c^G(\Delta_G^m) \subseteq l_\infty^G(\Delta_G^m)$. Hence $a \in [c^G(\Delta_G^m)]^\alpha$. This completes the proof. \square

Theorem 4.4. Let X stand for l_∞^G or c^G . Then

$$[X(\Delta_G^m)]^\alpha = \{a = (a_k) : {}_G \sum_k e^{k^m} \odot |a_k|^G < \infty\}.$$

Proof.

$$\begin{aligned} [l_\infty^G(\Delta_G^m)]^\alpha &= [Dl_\infty^G(\Delta_G^m)]^\alpha && \text{by Lemma (4.3)} \\ &= \{a = (a_k) : {}_G \sum_k e^{k^m} \odot |a_k|^G < \infty\} && \text{by Lemma (4.1)}. \end{aligned}$$

Again

$$\begin{aligned} [c^G(\Delta_G^m)]^\alpha &= [Dc^G(\Delta_G^m)]^\alpha && \text{by Lemma (4.3)} \\ &= [Dl_\infty^G(\Delta_G^m)]^\alpha && \text{by Lemma (4.2)} \\ &= \{a = (a_k) : {}_G \sum_k e^{k^m} \odot |a_k|^G < \infty\} && \text{by Lemma (4.1)}. \end{aligned}$$

\square

Corollary 4.5. For $X = l_\infty^G$ or c^G , we have

$$\begin{aligned} [X(\Delta_G)]^\alpha &= \{a = (a_k) : {}_G \sum_k e^k \odot |a_k|^G < \infty\}, \text{ and} \\ [X(\Delta_G^2)]^\alpha &= \{a = (a_k) : {}_G \sum_k e^{k^2} \odot |a_k|^G < \infty\}. \end{aligned}$$

Proof. Putting $m = 1$ and $m = 2$ in Theorem (4.4), the results follow. \square

Theorem 4.6. Let X stand for l_∞^G or c^G and $U_2 = \{a = (a_k) : \sup_k e^{k^{-m}} \odot |a_k|^G < \infty\}$. Then $[X(\Delta_G^m)]^{\alpha\alpha} = U_2$.

Proof. Let $a \in U_2$ and $x \in [X(\Delta_G^m)]^\alpha$, then by definition of U_2 and by Lemma (4.1), we get

$$\begin{aligned} {}_G \sum_k |a_k \odot x_k|^G &= {}_G \sum_k e^{k^m} \odot |x_k|^G \odot e^{k^{-m}} \odot |a_k|^G \\ &\leq {}_G \sum_k e^{k^m} \odot |x_k|^G \odot \sup_k e^{k^{-m}} \odot |a_k|^G < \infty. \end{aligned}$$

Hence $a \in [X(\Delta_G^m)]^{\alpha\alpha}$.

Conversely, let $a \in [X(\Delta_G^m)]^{\alpha\alpha}$ and $a \notin U_2$. Then we must have

$$\sup_k e^{k-m} \odot |a_k|^G = \infty.$$

Hence there exists a strictly increasing sequence $(e^{k(i)})$ of geometric integers[20], where $k(i)$ is a strictly increasing sequence of positive integers such that

$$e^{[k(i)]-m} \odot |a_{k(i)}|^G > e^{i^m}.$$

Let us define the sequence x by

$$x_k = \begin{cases} (|a_{k(i)}|^G)^{-1_G}, & k = k(i) \\ 1, & k \neq k(i). \end{cases}$$

where $(|a_{k(i)}|^G)^{-1_G}$ is the geometric inverse of $|a_{k(i)}|^G$ so that $|a_{k(i)}|^G \odot (|a_{k(i)}|^G)^{-1_G} = e$. Then we have

$${}_G \sum_k e^{k^m} \odot |x_k|^G = {}_G \sum_i e^{[k(i)]^m} \odot [|a_{k(i)}|^G]^{-1_G} \leq e^{i^{-m}} < \infty.$$

Hence $x \in [X(\Delta_G^m)]^\alpha$ and ${}_G \sum_k |a_k \odot x_k|^G = \sum e = \infty$. This is a contradiction as $a \in [X(\Delta_G^m)]^{\alpha\alpha}$. Hence $a \in U_2$. \square

Corollary 4.7. For $X = l_\infty^G$ or c^G , we have

$$[X(\Delta_G^2)]^{\alpha\alpha} = \{a = (a_k) : \sup_k e^{k-2} \odot |a_k|^G < \infty\}.$$

Proof. In Theorem (4.6), putting $m = 2$ we obtain the result. \square

Corollary 4.8. The sequence spaces $l_\infty^G(\Delta_G^m)$ and $c^G(\Delta_G^m)$ are not perfect.

Proof. Proof is trivial as $X^{\alpha\alpha} \neq X$ for $X = l_\infty^G(\Delta_G^m)$ or $c^G(\Delta_G^m)$. \square

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